

# Adaptive Optimal Nonparametric Regression and Density Estimation based on Fourier-Legendre expansion

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## Abstract

Motivated by finance and technical applications, the objective of this paper is to consider adaptive estimation of regression and density distribution based on Fourier-Legendre expansion, and construction of confidence intervals - also adaptive. The estimators are asymptotically optimal and adaptive in the sense that they can adapt to unknown smoothness.

Keywords:

Adaptive estimations, regression, density, martingale, confidence interval, Legendre polynomials.

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# 1 Introduction.

Among the latest fashions in nonparametric statistics are the so-called adaptive estimations (AE), i.e. estimations that use no apriory information about the estimated function. Many publications have recently appeared where AE are constructed which are optimal in order at a growing number of current observations on a continuum of various functional classes (cf. References for a list of works on AE, which does not, however, claim to be exhaustive).

In (Polyak B., at al., 1990), (Polyak B. at al., 1992), (Golubev G. at al., 1992) for instance, AE were constructed for the problem of estimating regression (R) which are optimal in order on many subspaces of space  $L_2$ , and non-adaptive confidence intervals were elaborated on the basis of the obtained estimations for the estimated regression function also in norm  $L_2$ , which later were somewhat improved in (Golubev at al., 1992).

In (Efroimovich S., 1985) AE were constructed for problem ( $D$ ) of estimating distribution density, which are optimal on ellipsoids in  $L_2$ .

In numerous publications by D. Donoho et al. (Donoho D at al., 1993(1), 1993(2), 1996, 1999(1), 1999(2) ) and in some others AE are constructed (and implemented) which are optimal in order on a number of Besov spaces. In those papers as well as in (Golybev G. at al., 1994), (Nussbaum M., 1985), (Tony Cai at al., 1999), (Lee G., 2003) diverse orthonormalized systems of functions are used to construct AE, such as wavelets, wedgelets, unconditional bases, splines, Demmler - Reinsch bases, Ridgelets (Candes E.J., 2003), (Dette H., 2003) etc.

The recent results about kernel estimations in the considered problems see, for example, (AAD W Van Der Vaart at al., 2003), Allal J., at al., 2003), (Corinne Berzin at al., 2003).

In (Ostrovsky E.I., 1996, 1997(1); 1997(2), 1999) AE were constructed on the basis of the trigonometric approximation theory.

**In this work we construct AE based on the orthogonal polynomial expansion series - the Legendre polynomials.**

**The AE proposed herein feature a speed of convergence which is optimal in order on any regular subspace compactly embedded in space  $L_2$ , the estimations are universal and very simple in form, which significantly facilitates their implementation; finally, we construct exponential adaptive confidence intervals (ACI), i.e. such that the tail of the confidence probability decreases with exponential speed.**

# 2 Problem statement. Denotations. Conditions.

**R.** Regression problem. Let  $f(x)$ ,  $x \in [-1, 1]$  be an unknown function, Riemann-integrable with a square and measured at points of the net  $x_i = x_{i,n} = -1 + 2i/n$ ,  $i = 1, 2, \dots, n$ ;  $n \geq 16$  with random independent centered:  $\mathbf{E}\xi_i = 0$  identically

distributed errors  $\{\xi_i\} : y_i = f(x_i) + \xi_i$ . It is required to estimate the function  $f(x)$  with the best possible precision from the values  $\{y_i\}$ .

**D. Estimation of distribution density.** On the basis of a set of independent identically distributed values  $\{\xi_i\}$ ,  $\xi_i \in [-1, 1]$ ,  $i = 1, 2, \dots, n$  it is required to estimate their common density  $f(x)$  (assumed to exist).

It is supposed that all the estimated functions  $f(\cdot) \in L_2[-1, 1]$ , therefore they are expanded in the norm of this space into a Fourier-Legendre series in the complete orthonormal system  $\{L_j(\cdot)\}$  on the set  $[-1, 1]$ :

$$f(x) = \sum_{j=0}^{\infty} c_j L_j(x); \quad c_j = \int_{-1}^1 L_j(x) f(x) dx,$$

where  $L_j(\cdot)$ ,  $j = 0, 1, 2, \dots$  are normalized Legendres polynomials. The Legendre polynomials are given by the Rodrigues formula:

$$P_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m} [(x^2 - 1)^m]$$

with orthogonal property:

$$I(k, m) \stackrel{\text{def}}{=} \int_{-1}^1 P_m(x) P_k(x) dx = 2/(2m + 1), \quad m = k,$$

otherwise  $I(k, m) = 0$ . We can define

$$L_k(x) = P_k(x) \sqrt{k + 0.5}.$$

Let us set  $\rho(N) = \rho(f, N) = \sum_{j=N+1}^{\infty} c_j^2$ . Evidently  $\lim_{N \rightarrow \infty} \rho(N) = 0$ . Let us also assume that only the non-trivial *infinite-dimensional case* will be considered, when an infinite multitude of Fourier coefficients  $f$  differs from zero, i.e.  $\forall N \geq 1 \Rightarrow \rho(N) > 0$ . Otherwise our estimations will converge in the sense  $L_2(\Omega \times [-1, 1])$  with speed  $1/\sqrt{n}$ .

Moreover, we assume further that (essentially infinite-dimensional case)

$$\lim_{N \rightarrow \infty} |\rho(N)| / \log N = +\infty. \quad (2.0).$$

In other words, the condition (2.0) means that there exists the constant  $q \in (0, 1)$  such that for all sufficiently great values  $N \Rightarrow \rho(N) \geq q^N$ .

The value  $\rho(N) = \rho(f, N)$  is known and is well studied in the *approximation theory*. Namely,  $\rho(f, N) = E_N^2(f)_2$ , where  $E_N(f)_p$  is the error of the best approximation of  $f$  by the algebraic polynomials of power not exceeding  $N$  in the  $L_p$  distance: for  $g : [-1, 1] \rightarrow R^1$  we will denote

$$\|g\|_p = \left( \int_{-1}^1 |g(x)|^p dx \right)^{1/p}, \quad p \in [1, \infty); \quad \|g\|_{\infty} = \sup_{x \in [-1, 1]} |g(x)|,$$

and closely connected with module of continuity of the form

$$\omega_{p,2}(f^{(k)}, \delta) = \sup_{h: |h| \leq \delta} |f^{(k)}(x+h) - 2f^{(k)}(x) + f^{(k)}(x-h)|_p,$$

for instance:

$$E_n(f)_p \leq C(p, r) n^{-r} \omega_{p,2}(f^{(r)}, 1/n).$$

[DeVore, Lorentz, p. 219-223]; arithmetical operations on the arguments of function  $f$  and their derivatives are understood as follows: at  $h > 0$   $x+h = \min(x+h, 1)$ ,  $x-h = \max(x-h, -1)$ ,  $h < 0$ .

Everywhere below condition  $(\gamma 1)$  will be considered fulfilled:

$$(\gamma 1) : \overline{\lim}_{N \rightarrow \infty} \rho(2N)/\rho(N) \stackrel{def}{=} \gamma < 1, \quad (2.1)$$

sometimes stronger conditions  $(\gamma)$  as well:

$$(\gamma) : \exists \lim_{N \rightarrow \infty} \rho(2N)/\rho(N) \stackrel{def}{=} \gamma < 1; \quad (2.2)$$

$$(\gamma 0) : \gamma = 0. \quad (2.3)$$

It is easy to show that it follows from condition (2.0)

$$\rho(N) \leq CN^{-2\beta}, \quad 2\beta \stackrel{def}{=} \log_2(1/\gamma) > 0. \quad (2.4)$$

In the problem (R) it will be assumed that  $\beta > 1/2$ . There are some grounds to suppose that at  $\beta < 1/2$  asymptotically optimal AE do not exist in the regression problem; for a similarly stated problem this was proved by Lepsky (Lepsky O., 1990).

Here and below the symbols  $C, C_r$  will denote positive finite constructive constants inessential in this context,  $\asymp$  is the usually symbol, in detail:

$$A(n) \asymp B(n) \Leftrightarrow C_1 \leq \liminf_{n \rightarrow \infty} A(n)/B(n) \leq$$

$$\limsup_{n \rightarrow \infty} A(n)/B(n) \leq C_2, \quad \exists C_1, C_2 \in (0, \infty).$$

the symbol  $A \sim B$  means that in the given concrete passage to the limit  $\lim A/B = 1$ .

**Examples.** Denote by  $W(C, \alpha, \beta)$  a class of functions  $\{f\}$  such that

$$\rho(f, N) \sim CN^{-2\beta}(\log N)^\alpha, \quad \exists C, \beta > 0; \alpha = \text{const};$$

$$W(\alpha, \beta) = \cup_{C>0} W(C, \alpha, \beta);$$

$$W(\beta) = W(0, \beta); \quad W = \cup_{\beta>0} W(\beta).$$

For the class of functions  $W$  the condition  $(\gamma)$  is fulfilled.

Also let us denote

$$Z(\alpha, \beta) = \{f : \rho(f, N) \sim \alpha \beta^N, \alpha > 0, \beta \in (0, 1);$$

and  $Z = \cup_{\alpha>0; \beta \in (0,1)} Z(\alpha, \beta)$ . For the functions of class  $Z$  the condition  $(\gamma_0)$  is also fulfilled. Besides, functions of class  $Z$  are analytical.

Denote for the problems **R**, **D** respectively at  $j < n$   $\hat{c}_j =$

$$(1/n) \sum_{i=1}^n y_j L_j(x_i); \quad \hat{c}_j = (1/n) \sum_{i=1}^n L_j(\xi_i); \quad (2.5)$$

$j = 0, 1, 2, 3, \dots, n-1$ ; and for the regression problem

$$B(n, N) = \sum_{k=N+1}^{2N} c_k(n)^2 + \sigma^2 N/n;$$

$\sigma^2 = \mathbf{Var}[\xi_i]$ ; for the problem (D) we define  $\sigma^2 = 1$  and

$$B(n, N) = \sum_{k=N+1}^{2N} c_k^2 + N/n;$$

and again for both the considered problems set  $B(n) =$

$$\min_{N=1,2,\dots,[n/3]} B(n, N), \quad N^0 = N^0(n) = \operatorname{argmin}_{N=1,2,\dots,[n/3]} B(n, N);$$

$$A(n, N) = \rho(N) + \sigma^2 N/n, \quad A(n) = \min_{N=1,2,\dots,[n/3]} A(n, N).$$

For instance, suppose that  $f \in W(C, \alpha, \beta)$ , then  $A(n) \asymp n^{-2\beta/(2\beta+1)} (\log n)^{\alpha/(2\beta+1)}$ , and in case  $f \in Z(\alpha, \beta) \Rightarrow A(n) \asymp \log n/n$ .

Our notation should not be surprising, as it follows from condition  $(\gamma_1)$  that all the introduced functionals  $\{B(n, N)\}$ ,  $\{B(n)\}$  arising from different problems are mutually  $\asymp$  equivalent. Besides, for the same reasons

$$A(n, N) \asymp B(n, N); \quad A(n) \asymp B(n).$$

Apart from that it is clear that in the regression problem conditions must be imposed not only on the estimated function, but on the measurement errors  $\xi_i$  too. We will consider here only the so-called *exponential* level. Indeed, we assume that in the regression problem the following condition is satisfied:

$$(Rq) : \exists q, Q \in (0, \infty), \Rightarrow \mathbf{P}(|\xi_i| > x) \leq \exp(-(x/Q)^q), x > 0.$$

The so-called classical projective estimates was introduced by N.N.Tchentsov [Tchentsov N.N., 1972, p. 286] (for the trigonometrical system instead considered

here Legendre's polynomials  $L_k(\cdot)$  will be considered as an estimates of the function  $f$ :

$$f(n, N, x) = \sum_{j=0}^N \hat{c}_j L_j(x). \quad (2.6)$$

Since, as is shown by Tchentsov,  $\mathbf{E} \|f(n, N, \cdot) - f(\cdot)\|^2 \asymp B(n, N)$ , the selection of the number of summands  $N$  optimal by order in the sense of  $L_2(\Omega) \times L_2[-1, 1]$  is given by the expression  $N = N^0(n)$  with the speed of convergence  $f(n, N^0, \cdot) \rightarrow f(\cdot)$  in the above-mentioned sense is  $\sqrt{A(n)}$ . I. A. Ibragimov and R. Z. Khasminsky (Ibragimov I., Khasminsky R., 1982) proved that no faster convergence exists on the regular classes of functions given by the value  $\sqrt{A(n)}$ .

However, the value  $\rho(f, N)$  or at least its order as  $N \rightarrow \infty$  are practically unknown as a rule. Below the adaptive estimation of  $f$  will be studied based only on the observations  $\{\xi_i\}$  and using no apriory information regarding  $f$ , and yet possessing the optimal speed of convergence at apparently weak restrictions. Set

$$\tau(N) = \tau(n, N) \stackrel{def}{=} \sum_{k=N+1}^{2N} \hat{c}_k^2, \quad N(n) \stackrel{def}{=} \underset{N \in (1, [n/3])}{\operatorname{argmin}} \tau(n, N), \quad (2.7)$$

$$\tau^*(n) = \min_{N \in (1, [n/3])} \tau(n, N),$$

*Our adaptive estimations  $\hat{f}$  in both considering problems have a universal view:*

$$\hat{f} = f(n, N(n), x) = \sum_{j=0}^{N(n)} \hat{c}_j L_j(x). \quad (2.8)$$

In case of a non-unique number of summands  $N(n)$  in (2.7) we choose the largest. Below the value  $N$  will always be *arbitrary* non-random integer number in the set of integers numbers of the segment  $1, 2, \dots, [n/3]$  and  $N(n)$  is the *random* variable defined in (2.7).

Note, that by using the Fast Legendre Transform technique described by D. Potts et al. (D. Potts et al., 1998), the amount of elementary operations for  $\hat{f}$  calculation is  $O(n \log n)$ , likewise in Fast Fourier Transform and in Fast Wavelet Transform.

Before proceeding to formulations and proofs let us clarify informally our idea for choosing  $N(n)$ . It is easy to find by direct calculation for the regression problem (and analogously for the problem D) that the coefficients estimations  $\hat{c}_k$  have a view:

$$\hat{c}_k = c_k(n) + n^{-1/2} \theta_k(n),$$

where  $asn \rightarrow \infty$

$$c_k(n) = n^{-1} \sum_{i=1}^n f(x_i) L_k(x_i) \rightarrow \int_{-1}^1 f(x) L_k(x) dx = c_k;$$

$$\theta_k(n) = n^{-1/2} \sum_{i=1}^n \xi_i L_k(x_i).$$

It follows from the multidimensional CLT that the variables  $\{\theta_k(n)\}$  as  $n \rightarrow \infty$  are asymptotically Gaussian distributed and independent:

$$\mathbf{Var}[\theta_k(n)] = n^{-1} \sum_{i=1}^n \sigma^2 L_k^2(x_i) \rightarrow \sigma^2 \int_{-1}^1 L_k^2(x) dx = \sigma^2;$$

$$\mathbf{E}\theta_k(n)\theta_l(n) = \sigma^2 n^{-1} \sum_{i=1}^n L_k(x_i)L_l(x_i) \rightarrow \sigma^2 \int_{-1}^1 L_k(x)L_l(x) dx = 0, \quad k \neq l.$$

Therefore, the variables  $\{\theta_k(n)\}$  are asymptotically independent and have approximately the normal distribution:

$$Law(\hat{c}_k) \asymp N(c_k, \sigma^2/n),$$

or equally

$$\hat{c}_k = c_k + \sigma \epsilon_k / \sqrt{n}, \quad Law(\epsilon_k) \asymp N(0, 1)$$

and also  $\{\epsilon_k\}$  are asymptotically independent. Therefore,  $\tau(n, N) \asymp$

$$\sum_{k=N+1}^{2N} c_k^2 + 2 n^{-1/2} \sigma \sum_{k=N+1}^{2N} c_k \epsilon_k + \sigma^2 n^{-1} \sum_{k=N+1}^{2N} \epsilon_k^2;$$

$$\mathbf{E}\tau(n, N) \asymp B(n, N), \quad \mathbf{Var}[\tau(n, N)] \asymp B(n, N)/n, \quad (2.9)$$

and therefore

$$N \rightarrow \infty, N/n \rightarrow 0 \Rightarrow \sqrt{\mathbf{Var}[\tau(n, N)]/\mathbf{E}\tau(n, N)} \rightarrow 0.$$

Note that in the case of the regression problem the condition

$$\beta > 1/2 \quad (2.10)$$

is essential which is common in statistical research (Polyak B. at al., 1990, 1992), (Lepsky O., 1990). We will assume in the problem (R) that the condition (2.10) is satisfied.

It follows from (2.9) that there are some grounds to conclude

$$\tau(n, N) \stackrel{a.s.}{\asymp} \mathbf{E}\tau(n, N) \asymp A(n, N)$$

and therefore

$$N(n) = \operatorname{argmin}_{N \leq n/3} \tau(n, N) \sim \operatorname{argmin}_{N \leq n/3} \mathbf{E}\tau(n, N) = N^0(n).$$

Also note that the number of summands  $N(n)$  proposed by us is a random variable (!) and that estimation (2.8) is non-linear by the totality of empirical Fourier coefficients  $\{\hat{c}_j\}$ .



### 3 Formulation of the main results.

Further we will investigate the exactness of our adaptive estimations in the  $L_2$  sense in our interval  $[-1,1]$  and will write as usually

$$\|f - g\|^2 = \int_{-1}^1 (f(x) - g(x))^2 dx.$$

Define also for the problem (R)

$$r = r(q) = 2q/(q + 4), q \in (0, 2); q \geq 2 \Rightarrow r = q/(q + 1), \quad (3.0)$$

and  $r = 1$  for the problem (D).

**Theorem R(q).** *Under the conditions (Rq),  $(\gamma 1)$ , in the problem **R** we propose that there exists a constant  $K_R = K_R(q, \gamma) \in (0, \infty)$  such that for the variable*

$$\zeta_R = \zeta_R(n) = B^{-1}(n) Q^{-2} \|\hat{f} - f\|^2 - Q^{-2} K_R$$

*the following inequality holds:*

$$\mathbf{P}(|\zeta_R| > u) \leq 2 \exp \left( -C u^{r/2} (n A(n))^{r/2} / \log \log n \right), \quad u > 1. \quad (3.1)$$

(See in comparison (Bobrov P. et al., 1997); here the exponent indices are significantly decreased.)

**Theorem D(q).** *Under the condition  $(\gamma 1)$ , in the problem **D** we propose that there exists a constant  $K_D = K_D(q, \gamma)$  such that for the variable*

$$\zeta_D = \zeta_D(n) = B^{-1}(n) \|\hat{f} - f\|^2 - K_D$$

*the following inequality holds:*

$$\mathbf{P}(|\zeta_D| > u) \leq 2 \exp \left( -u^{1/2} (n A(n))^{1/2} / \log \log n \right), \quad u > 1. \quad (3.2)$$

This result improves the one for the Fourier approximation of (Ostrovsky E., Sirota L., 2004).

**Theorem (Rq) a.s.** *If in the problem (R) under condition (Rq) for arbitrary  $\varepsilon > 0$  the series*

$$\sum_{\{n > 16\}} \mathbf{P}_n(\varepsilon) \stackrel{def}{=} \sum_{\{n > 16\}} \exp \left( -\varepsilon \frac{(n A(n))^{r/2}}{\log \log n} \right) < \infty, \quad (3.3)$$

*converges, then in the sense of convergence with probability one*

$$\lim_{n \rightarrow \infty} \tau^*(n)/B(n) = 1 \quad (3.4a)$$

*and*

$$\overline{\lim}_{n \rightarrow \infty} \|\hat{f} - f\|^2 / B(n) \leq K_R. \quad (3.4.b)$$

Let us make another additional assumption (v) with regard to the class of estimated functions  $\{f\}$ . Denote

$$H_n(v) = \inf_{N \in [N^0/v, N^0 v]} \frac{B(n, N)}{B(n)}, \quad v = \text{const} > 1;$$

$$\mathbf{P}^{(N)}(n, v) = \exp \left( -(\sqrt{H_n(v)} - 1)^{r/2} (nB(n))^{r/2} / \log \log n \right).$$

(At  $v \geq N^0$  the left interval is absent, at  $v \geq n/(3N^0)$  the right interval is absent.)  
*Condition (v) :*

$$\forall v > 1 \Rightarrow \sum_{n \geq 16} \mathbf{P}^{(N)}(n, v) < \infty.$$

The classes of functions satisfying conditions ( $\gamma 1$ ) and (v) will be called regular. Classes  $W$  and  $Z$  are regular.

**Corollary 1.** *If in addition to the conditions of theorem  $R(q)$  the condition (v) holds, then a.e.*

$$\lim_{n \rightarrow \infty} N(n)/N^0(n) = 1. \quad (3.5)$$

**Theorem (D) a.s.** *Let for problem (D) besides the above-formulated assumptions, condition (3.3) also be fulfilled with  $r/2$  replaced by  $1/2$ . Then the factual convergences of (3.4 a) and (3.4.b) are asserted here as well.*

**Corollary 2.** *Analogously if in addition to the our the condition (v) holds, then also with probability one*

$$\lim_{n \rightarrow \infty} N(n)/N^0(n) = 1.$$

#### 4. Proofs.

The proofs of the theorem (Rq) and (Dq) are similar to proofs of the our result in (Ostrovsky E., Sirota L., 2004) for trigonometrically approximation for  $f(x)$ ; we will use the known properties of Legendre's polynomials (Kallaev, 1970), (Szegő, 1959). For instance,

$$\sup_k \sup_{x \in [-1, 1]} |L_k(x)| (1 - x^2)^{1/4} < \infty.$$

Instead of the semi invariant estimations for polynomials from independent random variables (Saulis, Statuliavitchius, 1989) we will use the modern estimations for polynomial martingales (Hall, Heyde, 1980) [pp. 115 - 120], (Ostrovsky. E, 2004).

First of all we consider the problem of regression (R).

We will assume without loss of generality  $Q = 1$ .

STEP 1. Let us write the exact expression for the important variables. Introduce the notation:

$$\delta_k(n) \stackrel{def}{=} c_k(n) - c_k.$$

We can write:

$$\begin{aligned}\hat{c}_k(n) &= c_k + \delta_k(n) + n^{-1} \sum_{i=1}^n \xi_i L_k(x_i), \\ (\hat{c}_k(n))^2 &= c_k^2 + \delta_k^2(n) + \sigma^2 n^{-2} \sum_{i=1}^n L_k^2(x_i) + \\ &2c_k \delta_k(n) + 2n^{-1} \sum_{i=1}^n c_k \xi_i L_k(x_i) + 2n^{-1} \sum_{i=1}^n \delta_k(n) \xi_i L_k(x_i) + \\ &n^{-2} \sum_{i=1}^n (\xi_i^2 - \sigma^2) L_k^2(x_i) + n^{-2} \sum_{i \neq j} \xi_i \xi_j L_k(x_i) L_k(x_j).\end{aligned}$$

We have for the variables  $\tau(N, n)$  (and further for the variables  $\Delta^2 = \Delta^2(N, n) = \|\hat{f} - f\|^2$ ) :  $\tau(n, N) =$

$$\begin{aligned}&\left[ \sum_{k=N+1}^{2N} c_k^2 + 2 \sum_{k=N+1}^{2N} c_k \delta_k(n) + \sum_{k=N+1}^{2N} \delta_k^2(n) + \sigma^2 n^{-1} \sum_{k=N+1}^{2N} n^{-1} \sum_{i=1}^n L_k^2(x_i) \right] + \\ &\left[ 2n^{-1} \sum_{i=1}^n \xi_i \sum_{k=N+1}^{2N} \delta_k(n) L_k(x_i) + 2n^{-1} \sum_{i=1}^n \xi_i \sum_{k=N+1}^{2N} c_k L_k(x_i) \right] + \tau_2, \\ &\tau_2 = \left[ n^{-1} \sum_{i=1}^n (\xi_i^2 - \sigma^2) n^{-1} \sum_{k=N+1}^{2N} L_k^2(x_i) \right] + \\ &\left[ 2n^{-1} \sum_{1 \leq i < j \leq n} \xi_i \xi_j n^{-1} \sum_{k=N+1}^{2N} L_k(x_i) L_k(x_j) \right],\end{aligned}\tag{4.1}$$

where  $\tau = \tau_0 + \tau_1 + \tau_2$ ;  $\tau_m = \tau_m(n, N)$ ,  $\tau_0$  is the deterministic part of  $\tau : \mathbf{E}\tau = \tau_0 \sim B(n, N)$ ,  $\tau_1$  is the linear combination of  $\{\xi_i\}$ ,  $\tau_2$  is the bilinear combination of  $\{\xi_i\}$ .

It is easy to verify using the known properties of Legendre's polynomials that  $\tau_1 \asymp B(n, N)$  and that

$$\mathbf{Var}[\tau_1] \leq CB(n, N)/n, \quad \mathbf{Var}[\tau_2] \leq CB(n, N)/n.$$

STEP 2. Note that the sequences  $\eta_1(n) = \sum_{i=1}^n b(i) \xi(i)$ ,  $\eta_2(n) = \sum_{i=1}^n b(i) (\xi_i^2 - \sigma^2)$  and

$$\eta_3(n) = \sum_{1 \leq i < j \leq n} b(i, j) \xi_i \xi_j,$$

where  $\{b(i)\}, \{b(i, j)\}$  are a non-random sequences, with the second component  $F(n) = \sigma(\{\xi_i\}, i = 1, 2, \dots, n)$ , i.e.  $\{\eta_s(n), F(n)\}, s = 1, 2, 3; \{F(n)\}$  is the natural sequence (flow) of sigma-algebras, are martingales.

It follows from the main result of paper (Ostrovsky E., Sirota L., 2004), devoted to the exponential and moment estimations for martingale distributions, that

$$\sup_{n \geq 16} \mathbf{P}(|\tau_{1,2}|/\sqrt{\mathbf{Var}[\tau_{1,2}]} > x) \leq \exp(-Cx^r), \quad x > 0. \quad (4.2)$$

STEP 3. Now we intend to use on the basis of inequality (4.2) the Law of Iterated Logarithm (LIL) for the martingales (Hall, Heyde, 1980) [pp. 115 - 121] in the more convenient for us form (Ostrovsky, 1999) [pp. 79 - 83]. Namely, if we denote

$$\nu(n, N) = (\tau(n, N) - \mathbf{E}\tau(n, N))/\sqrt{B(n, N)/n},$$

then  $\mathbf{E}\nu(n, N) = 0$  and if we denote

$$\zeta \stackrel{\text{def}}{=} \sup_{n \geq 16} \sup_{N \in [1, n/3]} |\nu(n, N)|/\log \log n,$$

then  $\zeta < \infty$  a.e. and for the random variable  $\zeta$  we have for all positive values  $x, x > 0$  the tail inequality

$$\mathbf{P}(\zeta > x) \leq \exp(-C(q)x^r). \quad (4.3)$$

The inequality (4.3) may be rewritten as follows:

$$\tau(n, N) = \mathbf{E}\tau(n, N) + \zeta(n, N) \log \log n \sqrt{B(n, N)/n}, \quad (4.4)$$

where  $\zeta = \sup_{n, N} |\zeta(n, N)|$  satisfies the inequality (4.3),  $\mathbf{E}\tau(n, N) = B(n, N)(1 + \theta(n))$ ,  $\theta(n)$  is non-random and  $\lim_{n \rightarrow \infty} \theta(n) = 0$ .

STEP 4. Let  $M$  be some subset of an integer segment  $S = [1, 2, \dots, n]$ ,  $\overline{M} = S \setminus M$ ,  $\pi(M) \stackrel{\text{def}}{=} \mathbf{P}(N(n) \in M)$ , and assume that

$$v = v(n, M) \stackrel{\text{def}}{=} \inf_{N \in M} B(n, N)/B(n) > 1.$$

Then under conditions  $(\gamma)$  and  $(Rq)$

$$\pi(M) \leq 2 \exp\left(-C \left[(\sqrt{v} - 1) n B(n)\right]^{r/2} / (\log \log n)\right). \quad (4.5)$$

**Proof.** We obtain for the case of  $(Rq)$ , denoting  $\overline{v} = \max_{N \in S} |\nu(n, N)|$ :

$$\pi(M) = \mathbf{P}(N(n) \in M) = \mathbf{P}\left(\min_{N \in \overline{M}} \tau(n, N) > \min_{N \in M} \tau(n, N)\right) =$$

$$\mathbf{P}\left(\min_{N \in \overline{M}} (B(n, N) + \sqrt{B(n, N)/n} (\log \log n)^{1/r} \nu(n, N)) >$$

$$\min_{N \in M} \left( B(n, N) + \sqrt{B(n, N)/n} (\log \log n)^{1/r} \nu(n, N) \right) \leq$$

$$\mathbf{P}(B(n) + \sqrt{B(n)/n} (\log \log n)^{1/r} \bar{\nu} > vB(n) - \sqrt{vB(n)/n} (\log \log n)^{1/r} \bar{\nu}).$$

We find solving the inequality under the probability symbol relative to

$$\bar{\nu} : \pi(M) \leq \mathbf{P} \left( \bar{\nu} (1 + \sqrt{v}) \sqrt{B(n)/n} (\log \log n)^{1/r} \geq (v - 1)B(n) \right) \leq$$

$$\mathbf{P} \left( \bar{\nu} \geq \frac{v - 1}{\sqrt{v} + 1} \frac{\sqrt{nB(n)}}{(\log \log n)^{1/r}} \right) = \mathbf{P} \left( \bar{\nu} \geq (\sqrt{v} - 1) \frac{\sqrt{nB(n)}}{(\log \log n)^{1/r}} \right). \quad (4.6)$$

Using our estimations (4.6) for  $\tau$ , we arrive to the assertion (4.5).

Note that under our condition (2.0)

$$nB(n) > C \log n, \quad C_1 \in (0, \infty),$$

therefore under our conditions for all values  $v, v > 1$

$$\lim_{n \rightarrow \infty} \pi(M) = 0.$$

If in addition for any  $\varepsilon > 0$  the series  $\sum_n \mathbf{P}_n(\varepsilon)$  converges, the assertions (3.4.a),(3.4.b) and corollaries 1,2 to be proved follows from the lemma of Borel-Cantelli.

The rest is proved analogously if it is taken into account that  $N \geq N^0(n)(1 + \varepsilon)$ ,  $\varepsilon \in (0, 1]$  and condition (v) lead to the inequality  $B(n, N) \geq (1 + C\varepsilon^2)B(n)$ ,  $\varepsilon \in (0, 1)$ ; this completes the proof.

Analogously we can prove the theorem **(Rq)a.s.**, on the basis of inequality:

$$\mathbf{P}_n(\varepsilon) \leq \exp \left( -C\varepsilon^{r/2} (nA(n)^{r/2})/(\log \log n) \right).$$

Note in addition that at  $v > 2$

$$\tau^*(n) \leq B(n)(1 + \theta(n)) + \bar{\nu} \log \log n \sqrt{B(n)/n},$$

therefore we have for sufficiently great values  $n$  :

$$\mathbf{P}(|\tau^*(n)/(B(n)(1 + \theta(n)))| > v) \leq \exp \left( -C\varepsilon^{r/2} (nA(n)^{r/2})/(\log \log n) \right) \quad (4.7)$$

and analogously

$$\mathbf{P}(|\tau^*(n)/(B(n)(1 + \theta(n)))| < 1/v) \leq \exp\left(-C\varepsilon^{r/2} (nA(n)^{r/2})/(\log \log n)\right). \quad (4.8)$$

STEP 5. Let us consider here the main variables  $\Delta^2$ . We have:  $\Delta^2 =$

$$\begin{aligned} &= \left[ \sum_{k=N(n)+1}^{\infty} c_k^2 + \sigma^2 n^{-1} \sum_{k=0}^{N(n)} n^{-1} \sum_{i=1}^n L_k^2(x_i) + \sum_{k=0}^{N(n)} \delta_k^2(n) + 2 \sum_{k=0}^{N(n)} c_k \delta_k(n) \right] + \\ &\quad \left[ 2n^{-1} \sum_{i=1}^n \xi_i \sum_{k=0}^N \delta_k(n) L_k(x_i) + 2n^{-1} \sum_{i=1}^n \xi_i \sum_{k=0}^N c_k L_k(x_i) \right] + \\ &\quad \left[ n^{-1} \sum_{i=1}^n (\xi_i^2 - \sigma^2) n^{-1} \sum_{k=0}^{N(n)} L_k^2(x_i) + 2n^{-1} \sum_{1 \leq i < j \leq n} \xi_i \xi_j n^{-1} \sum_{k=0}^{N(n)} L_k(x_i) L_k(x_j) \right] = \end{aligned}$$

$\Delta_0 + \Delta_1 + \Delta_2$ . We have analogously to the investigation of the expression for  $\tau(n, N)$  using the condition  $\gamma$  for sufficiently large values  $n \geq n_0 > 2$ :

$$\begin{aligned} \|\hat{f} - f\|^2/B(n) &\leq CA(n, N(n))/B(n) + \Psi_3(N(n))/B(n) \leq \\ &C(1 - \gamma)^{-1} \tau^*(n)/B(n) + \Psi_3(N(n))/B(n) = \\ &C(1 - \gamma)^{-1} + (\tau^*(n)/B(n) - 1) + \Psi_3(N(n))/B(n), \end{aligned}$$

where, as can easily be seen,  $\Psi_3(N) = \Delta_1 + \Delta_2$ .

Then we will use the elementary inequality  $\mathbf{P}(\mathbf{A}) \leq \mathbf{P}(\mathbf{AB}) + \mathbf{P}(\overline{\mathbf{B}})$ , in which  $\mathbf{A}, \mathbf{B}$  are events. Setting  $\mathbf{A} =$

$$\{ \|\hat{f} - f\|^2/B(n) - C/(1 - \gamma) > u \}, \quad \mathbf{B} = \{ 1/v \leq \tau^*(n)/B(n) \leq v \},$$

we have at  $v \in (2, u - C)$ :

$$\mathbf{P}_0 \stackrel{\text{def}}{=} \mathbf{P}(\mathbf{AB}) \leq \mathbf{P}(v + \max_{N: 1/v < \tau^*/B(n) < v} |\Psi_3(N)|/B(n) > u).$$

We find analogously to the (Ostrovsky E., Sirota L., 2004):

$$\begin{aligned} &\mathbf{P}(v + \max_{N^0/v < N < vN^0} |\Psi_3(N)|/B(n) > u) \leq \\ &\exp\left(-C \frac{(u - v)^r}{v^{r/2}} \frac{((nA(n))^{r/2})}{\log \log n}\right). \end{aligned}$$

Thus,  $\mathbf{P}(A) \leq$

$$\exp\left(-C \frac{(u - v)^r}{v^{r/2}} \frac{((nA(n))^{r/2})}{\log \log n}\right) + \exp\left(-Cv^r \frac{((nA(n))^{r/2})}{\log \log n}\right).$$

Choosing  $v = 0.5u$  for sufficiently great values  $u$ ,  $u \geq C$ , we arrive to the assertion of theorem R(q).

**Remark 1.** Let us note, and use it below, a slight difference in the behaviors of the values  $\tau(n, N)$  and  $N(n)$  which consists in the peculiarity of condition (v). At  $v > 1$  we have (under the same conditions (Rq), (v)) :

$$\max \left( \mathbf{P} \left( \frac{N(n)}{N^0(n)} \leq \frac{1}{v} \right), \mathbf{P} \left( \frac{N(n)}{N^0(n)} > v \right) \right) \leq \exp \left( -Cv^r \frac{(nA(n))^{r/2}}{(\log \log n)} \right).$$

An analogous estimation for the probability  $\mathbf{P}(\tau^*(n)/B(n) > v)$  holds even without condition (v).

**Remark 2.** The *consistency* of the proposed estimations in the above-mentioned sense under all the introduced conditions, including (v), it follows from the assertions already proved. Indeed, since

$$A(n) \leq A(n, [\sqrt{n}]) \leq Cn^{-1/2} + \rho([\sqrt{n}]) \rightarrow 0,$$

then  $N^0(n) \rightarrow \infty$ ,  $N^0(n)/n \rightarrow 0$ , because otherwise the value

$$A(n) = A(n, N^0(n)) \asymp N^0(n)/n + \rho(N^0)$$

would not tend to zero.

Since  $N(n)/N^0(n) \rightarrow 1$ , then  $N(n) \rightarrow \infty$  and analogously  $N(n)/n \rightarrow 0$ , which proves the consistency of  $\hat{f}$ .

**We proceed now to the problem of estimating density (D).** Here

$$c_k = \int_{-1}^1 L_k(x) f(x) dx = \mathbf{E} L_k(\xi_i), \quad \hat{c}_k = n^{-1} \sum_{i=1}^n L_k(\xi_i),$$

$$\hat{f}(x) = \sum_{k=0}^{N(n)} \hat{c}_k L_k(x) = n^{-1} \sum_{i=1}^n \sum_{k=0}^{N(n)} L_k(\xi_i) L_k(x).$$

The functional  $\tau$  may be written as  $\tau(n, N) =$

$$n^{-1} \sum_{i=1}^n n^{-1} \sum_{k=N+1}^{2N} L_k^2(\xi_i) + 2n^{-2} \sum_{1 \leq i < j \leq n} \sum_{k=N+1}^{2N} L_k(\xi_i) L_k(\xi_j).$$

Let us denote

$$G_k(x, y) = \sum_{m=0}^k L_m(x) L_m(y).$$

It is known (Bateman H., Erdelyi A., 1953) [chapter 10, section 10], that if  $x \neq y$ , then

$$G_k(x, y) = (k+1) [P_{k+1}(x)P_k(y) - P_k(x)P_{k+1}(y)]/(x-y)$$

and

$$G_k(y, y) = (k + 1) \left[ P_k(y) P'_{k+1}(y) - P_{k+1}(y) P'_k(y) \right].$$

Also

$$(1 - x^2) P'_k(x) = k [P_{k-1}(x) - x P_k(x)].$$

Therefore, we have in the considered problem (D)

$$\begin{aligned} \tau(n, N) &= n^{-1} \sum_{i=1}^n n^{-1} (G_{2N}(\xi_i) - G_N(\xi_i)) + \\ &2n^{-2} \sum_{1 \leq i < j \leq n} (G_{2N}(\xi_i, \xi_j) - G_N(\xi_i, \xi_j)) = \end{aligned}$$

$\tau_0 + \tau_1 + \tau_2$ ,  $\mathbf{E}\tau = \tau_0 \sim B(n, N)$ , and the second (and the first) expression for the  $\tau$ , i.e.  $\tau_2$  is the so-called  $U$  statistics.

We find by direct calculation (as in the case of problem R):

$$\mathbf{Var}[\tau_1] \leq CB(n, N)/n, \quad \mathbf{Var}[\tau_2] \leq CB(n, N)/n.$$

Recall that the  $U$  statistic with correspondent sequence of sigma-algebras is also a martingale. Using the exponential boundaries for the martingale distribution, (Ostrovsky E., 2004), (Korolyuk B.S., Borovskich Yu., 1993) etc., we obtain:

$$\sup_{n \geq 16} \mathbf{P}(|\tau_{1,2}| / \sqrt{\mathbf{Var}[\tau_{1,2}]} > x) \leq \exp(-Cx), \quad x > 0,$$

$$\mathbf{P}(\zeta > x) \leq \exp(-Cx),$$

where  $\zeta = \sup_{n, N} |\zeta(n, N)|$ ,

$$\tau(n, N) \stackrel{def}{=} \mathbf{E}\tau(n, N) + \zeta(n, N) \log \log n \sqrt{B(n, N)/n}.$$

Repeating the considerations of (Ostrovsky E., Sirota L., 2004] we complete the proof.

**5. Adaptive confidence intervals (ACI).** Let us now describe the use of our results for the construction of ACI. Note first of all that the probability

$$\mathbf{P}_f(u) = \mathbf{P}(\|\hat{f} - f\|^2 > u)$$

with rather weak conditions in all the considered problems permits estimation of the form

$$\mathbf{P}_f(u) \leq 2 \exp\left(-\varphi(C, n, B(n))u^{r/2}\right) \stackrel{def}{=} \mathbf{P}_f^+(u), \quad u > C_1. \quad (5.1)$$



As it is proved above, the variables  $B(n), C, C_1$  have respective consistent estimates, for example,

$$B(n) \approx \min_{N \leq n/3} \tau(n, N) = \tau^*(n).$$

The values  $C, C_1$  also depends on  $\gamma$  and on the constants  $C_j$  appearing in the definition of conditions  $(\gamma), (v)$ . With very weak conditions they can also be estimated consistently by the sampling in the following way. Set  $M = M(n) = \lceil \exp(\sqrt{\log n}) \rceil$ ; then, if conditions  $(\gamma), (v)$  are fulfilled, a system of an asymptotic equalities can be written:

$$\begin{aligned} \tau(M) - \sigma_s M/n &\sim (1 - \gamma)\rho(M); \\ \tau(2M) - 2\sigma_s M/n &\sim \gamma(1 - \gamma)\rho(M); \\ \tau(4M) - 4\sigma_s M/n &\sim \gamma^2(1 - \gamma)\rho(M), \end{aligned}$$

where the symbol  $s$  denotes the number of problem.

Solving this system, we find the consistent (*mod*  $\mathbf{P}$ ) estimate of  $\gamma$ :

$$\hat{\gamma} = \frac{\tau(4M) - 2\tau(2M)}{\tau(2M) - 2\tau(M)}.$$

(The parameter  $\sigma_s$  can also be estimated consistently, but that is not necessary for us). The constants  $C_1, C$  also can be consistent determined.

Substituting the obtained estimates of all the parameters into (5.1), we get to the estimate of the confidence probability

$$\mathbf{P}_{\mathbf{f}}^+(u) \leq 2 \exp \left( -\phi(C(\hat{\gamma}, \hat{C}_1, \hat{C}), n, \tau^*(n)) u^{r/2} \right) \stackrel{def}{=} \hat{\mathbf{P}}_f(u). \quad (5.2)$$

then, equating the right-hand part of (5.2) of the unreliability of the confidence interval  $\delta$  to, say, the magnitude 0.05 or 0.01, we calculate  $u = u(\delta)$  from the relation

$$\hat{\mathbf{P}}_f(u(\delta)) = \delta$$

and obtain approximately the *adaptive confidence interval* for  $f$  reliability  $1 - \delta$  of the form

$$\|\hat{f} - f\|^2 \leq u(\delta) \min_{N \leq n/3} \tau(n, N). \quad (5.3)$$

But for a rough estimate of the error from replacing  $f$  by  $\hat{f}$  the following quite simple method can be recommended. Since

$$\frac{\|\hat{f} - f\|^2}{B(n)} = \frac{A(n, N(n))}{B(n)} + \frac{\Psi_3(N(n))}{B(n)}, \quad (5.4)$$

and the second term in the right-hand part of (5.4) a.s. tends to zero, while the first term, if conditions  $(\gamma), (v)$  are fulfilled, has  $1/(1 - \gamma)$  as its limit, we thus prove the following assertion apparently well known to specialists in nonparametric

statistics for non-adaptive estimation:

**Theorem c.i.** *If the following conditions are fulfilled in our problems: in the problem  $R$   $(Rq), (\gamma), (v)$  or  $(\gamma), (v)$  in problems  $D, S$ , then*

$$\overline{\lim}_{n \rightarrow \infty} \|\hat{f} - f\|^2 / B(n) \leq 1/(1 - \gamma). \quad (5.5)$$

In order to construct an adaptive confidence interval assertion (5.5) can be reformulated as follows. *With probability tending to 1 as  $n \rightarrow \infty$*

$$\|\hat{f} - f\|^2 \leq B(n)/(1 - \gamma), \quad (5.6)$$

and ACI is constructed by replacing the values  $B(n), \gamma$  by their consistent estimates:

$$\|\hat{f} - f\|^2 \leq \tau^*(n) \frac{\tau(2M) - 2\tau(M)}{3\tau(2M) - 2\tau(M) - \tau(4M)}. \quad (5.7)$$

A more exact result will be obtained by taking into account the following term of the expansion of the value  $\|\hat{f} - f\|^2$ :

$$\frac{\|\hat{f} - f\|^2}{B(n)} \leq \frac{1}{1 - \gamma} + \frac{\zeta}{\sqrt{N^0(n)}}(1 + \epsilon_n),$$

where  $\epsilon_n \rightarrow 0$ ;  $\mathbf{P}(|\zeta| > u) \leq 2 \exp(-Cu^{r/2})$  and  $C$  no longer depends on  $n$ . Equating the probability  $\mathbf{P}(|\zeta| > u)$ , more exactly its estimate  $2 \exp(-Cu^{r/2})$  to the value  $\delta$ ,  $\delta \approx 0+$ , we will easily find  $u = u(\delta)$  and construct an approximate ACI with reliability  $\approx 1 - \delta$  of the form

$$\|\hat{f} - f\|^2 \leq \frac{\tau^*(n)}{1 - \hat{\gamma}} + \tau^*(n)u(\delta).$$

Closer consideration reveals an effect that somewhat reduces the exactness of ACI. Let (as is true in all the three considered problems under the formulated assumptions)

$$\mathbf{P}(\|\hat{f} - f\|^2 / B(n) > u) \leq \exp(-\phi(C_1 u)), \quad \phi(u) = \phi(n, u),$$

$$\mathbf{P}(\tau^*(n)/B(n) < 1/u) \leq \exp(-\phi(C_2 u)), \quad u > C,$$

where at  $u \rightarrow \infty \Rightarrow \phi(u) \rightarrow 0$ . We denote

$$\mathbf{Q}(u) = \mathbf{P}(\|\hat{f} - f\|^2 / \tau^*(n) > u).$$

**Theorem  $\tau$ .** *At  $u \leq C/B(n)$  the following inequality holds:*

$$Q(u) \leq 2 \exp(-\phi(C\sqrt{u})).$$

**Proof.** We have by the full probability formula we (we will understood  $\mathbf{P}(A/B)$  as the conditional probabilities, if, of course,  $A$  and  $B$  are events):

$$\begin{aligned} \mathbf{Q}(u) &\leq \mathbf{P} \left( \frac{\|\hat{f} - f\|^2}{\tau^*(n)} > u / \frac{\tau^*(n)}{B(n)} > \frac{1}{v} \right) \cdot \mathbf{P} \left( \frac{\tau^*(n)}{B(n)} > \frac{1}{v} \right) + \\ &+ \mathbf{P} \left( \frac{\|\hat{f} - f\|^2}{\tau^*(n)} > u / \frac{\tau^*(n)}{B(n)} \leq \frac{1}{v} \right) \cdot \mathbf{P} \left( \frac{\tau^*(n)}{B(n)} \leq \frac{1}{v} \right) \stackrel{def}{=} Q_1 + Q_2; \\ Q_1 &\leq \mathbf{P} \left( \|\hat{f} - f\|^2 / B(n) > u/v \right) \leq \exp(-\phi(C_1 u/v)); \\ Q_2 &\leq \mathbf{P} (\tau^*(n)/B(n) \leq 1/v) \leq \exp(-\phi(C_2 v)). \end{aligned}$$

Summing up and put  $v = C_3 \sqrt{u}$ , we obtain the assertion of the theorem.

The increase in the probability  $\mathbf{Q}$  compared to  $\mathbf{P}_f$  is apparently explained by the ability of the denominator, i.e.  $\tau^*(n)$  to take values nearly to zero.

Note in conclusion that the estimates proposed by us have successfully passed experimental tests on problems **R**, **D** by simulate of modeled with the use of pseudo-random numbers as well as on real data (of financial data) for which our estimations of the regression and density were compared with classical estimates obtained by the kernel and wavelets estimations method. The precision of estimations proposed here is better.

The advantage of our estimations in comparison to the trigonometrical estimations [Ostrovsky, Sirota, 2004] is especially in the case when the estimating function  $f(\cdot)$  is not periodical.

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